Towards a classification of CMC-1 Trinoids in hyperbolic space via conjugate surfaces*

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Abstract

We derive necessary conditions on the parameters of the ends of a CMC-1 trinoid in hyperbolic 3-space $\mathbb{H}^3$ with symmetry plane by passing to its conjugate minimal surface. Together with [Dan03], this yields a classification of generic symmetric trinoids. We also discuss the relation to other classification results of trinoids in [BPS03] and [UY00].

To obtain the result above, we show that the conjugate minimal surface of a catenoidal CMC-1 end in $\mathbb{H}^3$ with symmetry plane is asymptotic to a suitable helicoid.

1 Introduction

A minimal surface in $\mathbb{R}^3$ can be presented by its *Weierstrass data*, i.e. as a map $\Phi_W : \Sigma \to \mathbb{R}^3$, where Σ is a Riemann surface, and Φ_W depends on (g, ω), a meromorphic function and a holomorphic 1-form on Σ.

Given a minimal surface, one can consider its *associate (minimal) surface*, which is determined by the Weierstrass data $(g, i\omega)$.

Bryant found a representation of constant mean curvature 1 (CMC-1) surfaces in $\mathbb{H}^3$ depending on the same data (see [Bry87]). Therefore, we call CMC-1 surfaces in $\mathbb{H}^3$ *Bryant surfaces*. A Bryant surface has a *minimal cousin*, the minimal surface determined by the same data (g, ω).

Given a Bryant surface, we define its *conjugate (minimal) surface* to be the associate minimal surface of its minimal cousin.

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Under this construction, a principal geodesic (i.e. a geodesic which is also a curvature line) on the Bryant surface corresponds to a straight line on its conjugate surface.

We define $I := (-\frac{1}{4}, \infty)\backslash\{0\}$, and introduce helicoids H_λ, catenoids C_λ^W, and catenoid cousins C_λ parametrized by $\lambda \in I$, such that:
The helicoid H_λ is the associate minimal surface of C_λ^W, and C_λ^W is the minimal cousin of the Bryant surface C_λ.

It is known that an end of a Bryant surface is asymptotic to some catenoid cousin or to a horosphere ([CHR01, Thm. 10]).

Clearly, one would expect that the conjugate surface of a catenoidal Bryant end is asymptotic to a suitable helicoid. However, this is not immediate, since the Bryant cousin relation is given by a second-order description only. For the similar situation of relating CMC-1 surfaces in $\mathbb{R}^3$ to minimal surfaces in S^3, there exists a first-order description. Using this, it is possible to conclude that asymptotics is preserved in this case (see [GKS01]).

For our situation, we show in section 3 that if a catenoidal end has a symmetry plane, then the asymptotics is indeed preserved:

Theorem 1.1. *Let E' be a symmetric Bryant end asymptotic to C_λ for some $\lambda \in I$. Then the conjugate minimal surface E'^c is asymptotic to H_λ.*

In section 4, we turn our attention to CMC-1 *trinoids* in $\mathbb{H}^3$. I.e., we examine Bryant surfaces of genus zero with three ends, all of which are catenoidal.
We study *symmetric* trinoids, i.e. trinoids which have a symmetry plane (determined by the asymptotic boundary points of their ends).

It follows from the classification by [UY00] that every (generic) trinoid is symmetric; since we present a different approach to this moduli problem, we do not use this result. One should look for a direct geometric proof that every properly immersed CMC-1 surface in $\mathbb{H}^3$ of genus zero and three ends has a symmetry plane.

A symmetric trinoid can be cut open along its symmetry plane to obtain two simply connected pieces. The conjugate surface of such a piece is a minimal surface bounded by three lines. Surfaces of this kind were already examined by Riemann (see [Rie61, sec. 17] or [Dar87]).

Using Theorem 1.1, this yields a necessary condition on the parameters of a generic trinoid:

Let $J := (0, \infty)\backslash\{\pi\}$; for a real number φ, we call

$$r(\varphi) := \min_{n\in\mathbb{Z}} |\varphi + 2n\pi|$$

the *reduced angle* of φ. Furthermore, let $\mathcal{T}$ be the set of interior points of the tetrahedron with vertices $(\pi, 0, 0)$, $(0, \pi, 0)$, $(0, 0, \pi)$, (π, π, π). Then we have:

Theorem 1.2. *If there exists a symmetric trinoid corresponding to the parameter triple $(\varphi_1, \varphi_2, \varphi_3) \in (J \backslash \pi\mathbb{Z})^3$, then for the triple of reduced angles holds (in the generic case):*

$$(r(\varphi_1), r(\varphi_2), r(\varphi_3)) \in \mathcal{T}.$$

On the other hand, minimal surfaces bounded by three lines are constructed in [Dan03]. His main result is:

Theorem 1.3 ([Dan03, Thm. 49]). *Let $(\varphi_1, \varphi_2, \varphi_3) \in (J \backslash \pi\mathbb{Z})^3$, and assume that $(r(\varphi_1), r(\varphi_2), r(\varphi_3))$ lies in $\mathcal{T}$.*

Under a certain polynomial condition (in the φ_i), there is a corresponding symmetric trinoid which arises from a minimal disk bounded by three lines.

In section 6, we compare the conditions given by the theorems above to the conditions found in [BPS03] and [UY00], and find that they are essentially the same:

Corollary 1.4. *The conditions of Theorems 1.2 and 1.3 are equivalent to those given by [BPS03].*

For symmetric parameter triples $(\varphi, \varphi, \varphi)$ with $\varphi \in (\pi/3, \pi)$, one can construct the minimal surface using a sequence of Plateau solutions, and show that it corresponds to a trinoid; for details, see [Bal03].

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2 Preliminaries

In this section, we present the material from the beginning of section 1 in more detail.

First, we recall the Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ and the Bryant representation for CMC-1 surfaces in $\mathbb{H}^3$:

The Weierstrass representation Theorem says that every minimal surface can be conformally parametrized as

$$\Phi_W(z) = \mathrm{Re}\oint_{z_0}^{z} \left((1-g^2)\omega, i(1+g^2)\omega, 2g\omega\right), \tag{1}$$

where z is in Σ, the parametrizing Riemann surface (possibly with boundary), and g (resp. ω) is a meromorphic function (resp. a holomorphic 1-form) on Σ. Furthermore, g has a pole of order k in z if and only if ω has a zero of order $2k$ in z.

The function g has a geometric meaning: it is the stereographic projection of the Gauss (or normal) map of the minimal surface Φ_W.

The pair (g, ω) is called the *Weierstrass data* of Φ_W.

Conversely, Weierstrass data on a Riemann surface Σ defines a minimal immersion from the universal cover $\tilde{\Sigma}$ into $\mathbb{R}^3$ via (1).

To a minimal surface Φ_W with Weierstrass data (g, ω), one has its *associate surface* $\bar{\Phi}_W$, which is given by the Weierstrass data $(g, i\omega)$; it turns out that Φ_W and $\bar{\Phi}_W$ are (locally) isometric. Note that if Φ_W is defined on Σ, it may happen that $\bar{\Phi}_W$ is defined on $\tilde{\Sigma}$ only.

For more details on the Weierstrass representation, we refer the reader to [Oss86, §8].

In his seminal paper [Bry87], Bryant showed that there is a representation of CMC-1 surfaces in $\mathbb{H}^3$ using exactly the same data as the Weierstrass representation. Thus, to a minimal surface Φ_W with Weierstrass data (g, ω), one obtains a *Bryant cousin* Φ_B; vice versa, every CMC-1 surface in $\mathbb{H}^3$ has a *minimal cousin*. The surfaces Φ_W and Φ_B are (locally) isometric, and their Gauss maps agree.

Definition 2.1. Following Rosenberg, we define a *Bryant surface* to be an immersed CMC-1 surface in $\mathbb{H}^3$.

Definition 2.2. Given a simply connected Bryant surface M, we define its *conjugate surface* M^c as the associate surface of M's minimal cousin (in $\mathbb{R}^3$).

Since the Bryant relation is a special case of Lawson's correspondence ([UY92]), a principal geodesic (i.e. a geodesic which is also a curvature line) corresponds to a principal geodesic under the Bryant cousin relation. Under the associate construction, principal geodesics go to straight lines. Thus principal geodesics on M are mapped to straight lines by M^c.

Example 2.3. We introduce our notation for the helicoids, the catenoids, and the catenoid cousins:

Parametrize the surfaces by $\Sigma = \mathbb{C}$: For $0 \neq \lambda \in \mathbb{R}$, the catenoid C_λ^W is the minimal surface with Weierstrass data $g = \exp(z)$, $\omega = \lambda \exp(-z)dz$, and the helicoid H_λ is its associate surface, with Weierstrass data $g = \exp(z)$, $\omega = \lambda i \exp(-z)dz$.

The formula for H_λ is

$$H_\lambda(x+iy) = 2\lambda \begin{pmatrix} \sinh x \sin y \\ -\sinh x \cos y \\ -y \end{pmatrix}. \tag{2}$$

If $\lambda \in I$, where $I := (-\frac{1}{4}, \infty)\backslash\{0\}$, we call the Bryant cousin of C_λ^W a *Catenoid Cousin* C_λ. Formulas for catenoid cousins C_λ in the upper halfspace model ($\mathbb{H}^3 = \{(u+iv, w) \,|\, u, v \in \mathbb{R}, w > 0\}$) are given in [Ros02, sec. 11]: The surfaces are again parametrized by $\mathbb{C}$; every line with constant imaginary part parametrizes a principal geodesic from the end of C_λ at 0 to the end at ∞ in $\mathbb{H}^3$. Set $a := \sqrt{1+4\lambda}$; then the formula for $C_\lambda(x+iy)$ is given by

$$\begin{aligned} u + iv &= \frac{-\lambda(e^x + e^{-x})e^{ax}}{\left(\frac{1}{2} + \lambda - \frac{1}{2}a\right)e^{-x} + \left(\frac{1}{2} + \lambda + \frac{1}{2}a\right)e^x} e^{iay} \\ w &= \frac{ae^{ax}}{\left(\frac{1}{2} + \lambda - \frac{1}{2}a\right)e^{-x} + \left(\frac{1}{2} + \lambda + \frac{1}{2}a\right)e^x} \end{aligned}$$

Note that the parametrization of C_λ is periodic with period $\frac{2\pi i}{\sqrt{1+4\lambda}}$.

Let $J := (0, \infty)\backslash\{\pi\}$, and define the bijective function $\tilde{\varphi} : I \to J$ by $\tilde{\varphi}(\lambda) := \frac{\pi}{\sqrt{1+4\lambda}}$.

We remark that the Catenoids (and Catenoid Cousins) $C_\lambda^{(W)}$ can alternatively be described by the Weierstrass data $g = z^\alpha, \omega = \frac{1-\alpha^2}{4\alpha} z^{-1-\alpha}$ on $\mathbb{C}^*$, where $\pi\alpha = \tilde{\varphi}(\lambda)$; see [ST01, Ex. 1.5].

3 Symmetric catenoidal Bryant ends and their conjugate surfaces

In this section, we show that the conjugate minimal surface of a catenoidal Bryant end with a symmetry plane is asymptotic to the corresponding helicoid.

Definition 3.1. An *annular Bryant end* is a Bryant surface with domain $\{0 < |z| \leq 1\}$ (or equivalently any other punctured disk with boundary).

Recall that a properly embedded Bryant annular end in $\mathbb{H}^3$ is asymptotic to some catenoid cousin or to a horosphere [CHR01, Thm. 10].

Definition 3.2. An annular Bryant end is called *catenoidal* if it is properly embedded and asymptotic to a catenoid cousin.

An annular Bryant end is called *symmetric* if it is properly embedded and has a symmetry plane.

Definition 3.3. A *minimal end bounded by rays* is a properly immersed minimal surface in $\mathbb{R}^3$ with domain $\{0 < |z| \leq 1, \operatorname{Im} z \geq 0\}$, such that $[-1, 0)$ and $(0, 1]$ are mapped to (monotonically parametrized) rays.

For a real number φ, we call $r(\varphi) := \min_{n\in\mathbb{Z}} |\varphi + 2n\pi|$ the *reduced angle* of φ.

The following Lemma is a slight generalization of [Dan03, L. 7]:

Lemma 3.4. *Let X be a minimal end bounded by horizontal rays with vertical limit normal for $z \to 0$. Assume that X is contained in a vertical slab (i.e. the vertical component of X is bounded), that the stereographic projection g of the Gauss map of X satisfies $g \sim z^\alpha$ for $z \to 0$ with $0 < \alpha \neq 1$, and that the vertical components of the boundary rays are $|\tilde{\varphi}^{-1}(\pi\alpha)|\pi\alpha$ apart. Then X is asymptotic to (part of) IH_λ for $\lambda = \tilde{\varphi}^{-1}(\pi\alpha)$ and some orientation preserving isometry I of $\mathbb{R}^3$.*

Proof. First we note that we can conclude from the proof of [Dan03, L. 7] that the angle between the boundary rays is the reduced angle $r(\pi\alpha)$. Additionally, observe that the (vertical) distance of the boundary rays is by assumption the distance of two lines in H_λ, where one has to be rotated by angle $\pi\alpha$ *in* H_λ to be mapped to the other one (cf. formula (2)).

If the boundary rays are not parallel, the claim is just [Dan03, L. 7]. The case of parallel boundary rays is not covered there; however, its proof still works in this case by our assumptions on the limit normal, X being contained in a slab, and the vertical distance of the rays. □

Proof of Theorem 1.1. It suffices to consider one symmetric piece E of E' bounded by principal geodesics (the curves of intersection with the symmetry plane). Let E^c denote the conjugate surface of this half. We assume E^c to be parametrized by $D := \{0 < |z| \leq 1, \operatorname{Im} z \geq 0\}$. By [CHR01], E' has a well-defined limit normal, which we may assume to be vertical.

Then E^c also has a vertical limit normal, so it is a minimal end bounded by horizontal rays. We show that E^c is contained in a vertical slab:

By [ST01], we may assume the Weierstrass data of E' to be of the form

$$g = z^\alpha(g_0 + zg_1(z)), \quad \omega = z^{-1-\alpha}(w_0 + zw_1(z))$$

with $g_0, w_0 \in \mathbb{C}$ such that $g_0 w_0 = \frac{1-\alpha^2}{4\alpha}$, and holomorphic functions g_1, w_1 on $\{|z| \leq 1\}$ (where $\pi\alpha = \tilde{\varphi}(\lambda)$, in particular $0 < \alpha \neq 1$).

Choose $z_0 \in (0,1] \subset D$; the third component of E^c is the negative of the imaginary part of the following integral:

$$\begin{aligned}\oint_{z_0}^{z} 2g\omega &= 2\oint_{z_0}^{z} \xi^{-1}\Big(g_0w_0 + \xi\big(g_0w_1(\xi) + w_0g_1(\xi) + \xi w_1(\xi)g_1(\xi)\big)\Big)d\xi \\ &= \frac{1-\alpha^2}{2\alpha}\oint_{z_0}^{z} \xi^{-1}d\xi + 2\underbrace{\oint_{z_0}^{z} g_0w_1(\xi) + w_0g_1(\xi) + \xi w_1(\xi)g_1(\xi)d\xi}_{=:C(z)}\end{aligned}$$

Hence, E^c is contained in a vertical slab, since C is bounded on D and the first summand corresponds to the third component of H_λ. Observe that the imaginary part of the first summand above is 0 for $z \in (0,1]$ and constant for $z \in [-1,0)$. We show that $\operatorname{Im} C(z) = 0$ for $z \in [-1,0)\cup(0,1]$: This is clear for $z \in (0,1]$, since $z_0 \in (0,1]$, and a horizontal ray is parametrized. Similarly, $\operatorname{Im} C(z) \equiv C_2$ for $z \in [-1,0)$ since this parametrizes another horizontal ray. Thus $\operatorname{Im} \oint_{1/n}^{-1/n} g_0w_1(\xi)+w_0g_1(\xi)+\xi w_1(\xi)g_1(\xi)d\xi$ is constant (i.e. independent from n and the path in D from $\frac{1}{n}$ to $-\frac{1}{n}$) and we have

$$C_2 = \operatorname{Im}\lim_{n\to\infty}\oint_{1/n}^{-1/n} g_0w_1(\xi) + w_0g_1(\xi) + \xi w_1(\xi)g_1(\xi)d\xi = 0.$$

This shows that the two boundary rays of E^c have positive vertical distance, which is equal to the distance of corresponding lines on H_λ. Now the conclusion follows via Lemma 3.4. □

Corollary 3.5. *Let E' be a symmetric Bryant end which is asymptotic to C_λ, and let E be a symmetric piece of E' as above. If $\varphi := \tilde{\varphi}(\lambda) \notin \pi\mathbb{Z}$, we have: The boundary rays l_1, l_2 of E^c are contained in IH_λ for some orientation-preserving isometry I of $\mathbb{R}^3$. In particular, the angle between the ends of l_1 and l_2 is $r(\varphi)$. The distance of these two lines is $h(\varphi) := |\lambda|\varphi$.*

Proof. First we note that $h : J \to \mathbb{R}$ is well-defined, because $\tilde{\varphi}$ is a bijective function (in fact $h(\varphi) = |\frac{\pi^2}{4\varphi} - \frac{\varphi}{4}|$).

The claim follows immediately from the proof of Theorem 1.1 and the formulas for helicoids. □

In case $\tilde{\varphi}(\lambda) \in \pi\mathbb{Z}$, the boundary rays are parallel, and we have a lower bound on their distance (by the distance of parallel lines in the corresponding helicoid).

4 Trinoids

Definition 4.1. We define a *trinoid* to be a properly immersed Bryant surface of genus zero with three ends, all of which are catenoidal. A *symmetric trinoid* is a trinoid T which has a symmetry plane P such that the asymptotic endpoints of T are contained in the asymptotic boundary of P.

Denote by $\mathcal{M}$ the space of symmetric trinoids with ends marked by 1, 2, 3, up to isometry (respecting the marks of the ends).

Observe that the symmetry plane P is uniquely determined if the asymptotic endpoints are distinct.

Pictures of trinoids can be found at
`http://www-sfb288.math.tu-berlin.de/~bobenko/Trinoid/webimages.html`; see also [BPS03].

Definition 4.2. We can define the map $\Psi : \mathcal{M} \to J^3$ sending a trinoid to the triple $(\varphi_1, \varphi_2, \varphi_3) \in J^3$, where $\varphi_i = \tilde{\varphi}(\lambda_i)$, and λ_i is the parameter of end i.

Lemma 4.3. *Any properly embedded Bryant surface M of genus zero with three ends is a symmetric trinoid.*

Proof. By Theorem [CHR01, Thm. 12], every end is catenoidal, and by [CHR01, Thm. 11], the three asymptotic boundary points are distinct and M is a bigraph over the plane containing them. □

We expect that the Lemma above generalizes to Alexandrov-embedded Bryant surfaces.

Note that a trinoid is a map $S^2 \backslash \{x_1, x_2, x_3\} \to \mathbb{H}^3$, where x_1, x_2, x_3 are distinct and correspond to the ends 1, 2, 3 respectively.

Lemma 4.4. *For a symmetric trinoid $M \in \mathcal{M}$, there is a unique principal geodesic of M joining x_1 to x_2, which we denote by l_{12}. Similarly, there is a unique principal geodesic l_{23} joining (x_2, x_3) and a unique principal geodesic l_{31} joining (x_1, x_2).*
Considering l_{12}, l_{23}, l_{31} as subsets of S^2, we have that $S^2 \backslash (l_{12} \cup l_{23} \cup l_{31} \cup \{x_1, x_2, x_3\})$ consists of exactly two components.

Proof. It is known that a principal geodesic is contained in a plane of symmetry of M (cf. [ST01, Prop. 3.2]). We conclude that the three lines we are looking for need to be contained in P, the symmetry plane of M from the definition.

Consider the graph G in S^2 with vertices $V := \{x_1, x_2, x_3\}$, and edges the principal geodesics of M contained in P which start or end in V (observe that both asymptotic ends of such a principal geodesic are in V).

Since every end of M is embedded, every vertex has degree two. Edges cannot intersect: Tangential contact is excluded by uniqueness of geodesics, and transversal intersection is impossible since M intersects P orthogonally near every point of G.

Thus, G consists of one, two, or three loops in S^2. Reflection in P maps every component of $S^2 \backslash G$ to an other component. Since all elements of V are fixed points of this reflection, G consists of one loop only. □

Corollary 4.5. *Consider a symmetric trinoid M and its symmetry plane P. Then there is a neighborhood N of $l_{12} \cup l_{23} \cup l_{31} \cup \{x_1, x_2, x_3\}$ in S^2 such that $M(N) \cap P = M(l_{12} \cup l_{23} \cup l_{31})$. In particular: Near its boundary, each component of $S^2 \backslash (l_{12} \cup l_{23} \cup l_{31} \cup \{x_1, x_2, x_3\})$ is mapped to a component of $\mathbb{H}^3 \backslash P$.*

If M is embedded, each component of $S^2 \backslash (l_{12} \cup l_{23} \cup l_{31} \cup \{x_1, x_2, x_3\})$ is mapped into a halfspace of $\mathbb{H}^3 \backslash P$. □

Given a symmetric trinoid M, we can (by an orientation-preserving isometry) assume that its symmetry plane is the equatorial plane $E = \{x_3 = 0\}$ of the Poincaré disk model (lying inside $\mathbb{R}^3$). Further, we can assume that the ends are marked increasingly if one looks from above (i.e. the direction of positive x_3).

Definition 4.6. Given a symmetric trinoid M, we divide its domain $S^2 \backslash \{x_1, x_2, x_3\}$ into two components along l_{12}, l_{23}, l_{31}, and we define M^+ to be the restriction of M to the closure of the component which is mapped to the upper half space near l_{12}, l_{23}, l_{31}, if M is put in the Poincaré model in the way explained above.

So M^+ is a map $\bar{D} \backslash \{x_1, x_2, x_3\} \to \mathbb{H}^3$, where x_1, x_2, x_3 are distinct points in ∂D (and D is the closed unit disk).
We choose the orientation on D and its boundary as depicted in Figure 1.

Note that M^+ is well-defined up to orientation-preserving hyperbolic isometries leaving the upper half-space in the Poincaré disk model invariant.

Define $M^c := (M^+)^c$ to be the conjugate minimal surface of M^+.

Since principal geodesics on a Bryant surface correspond to straight lines on its conjugate minimal surface, we have:

Lemma 4.7. *Let M be a symmetric trinoid. Then M^c is a minimal surface bounded by three straight lines.* □

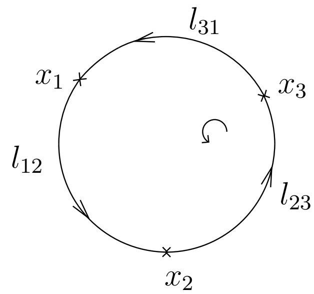

Figure 1: The domain of M^+ and M^c.

We mention another interesting fact about M^+:

Proposition 4.8. *If M^+ is embedded, then so is M.*

Proof. Assume the symmetry plane of M to be the equatorial plane E as before. We apply the Alexandrov-reflection technique (see, for example, [LR85]): Using a (continuous) family of planes which foliate the upper half-space, we conclude that the normal of M^+ at any point $p \in M^+ \cap E$ has non-positive vertical coordinate.

Similarly, we use a family of planes foliating the lower half-space to find that for a point $p \in M^+ \cap E$, the normal has to have non-negative vertical coordinate.

Thus, every component of $M \cap E$ is a principal geodesic, i.e. a curve of planar reflection (by [ST01, Prop. 3.2]). So M^+ is cut off wherever it reaches E (observe that there are no closed principal geodesics since M has genus zero), and it does not intersect the lower half-space; so M is embedded. $\square$

5 Necessary conditions on the constellation of boundary lines

In this section, we use the information about the constellation of lines which bound M^c to obtain a necessary condition on the parameter triple $\Psi(M)$ in the generic case.

Definition 5.1. A triple $(\lambda_1, \lambda_2, \lambda_3) \in I^3$ is called a *parameter triple*.

A triple of oriented lines (l_{12}, l_{23}, l_{31}) in $\mathbb{R}^3$ is called *admissible constellation* if

(i) There exists a parameter triple $(\lambda_1, \lambda_2, \lambda_3) \in I^3$ and orientation-preserving isometries I_1, I_2, I_3 of $\mathbb{R}^3$ such that

$$\{l_{12}, l_{31}\} \subset I_1(H_{\lambda_1}), \quad \{l_{23}, l_{12}\} \subset I_2(H_{\lambda_2}), \quad \text{and } \{l_{31}, l_{23}\} \subset I_3 H_{\lambda_3}.$$

(ii) Rotating $l_{i(i+1)}$ inside $I_i(H_{\lambda_i})$ maps $l_{i(i+1)}$ to $l_{(i+2)i}$ with the opposite orientation (for $i \in \mathbb{Z}_3$).

(iii) The distance of $l_{i(i+1)}$ and $l_{(i+2)i}$ is $h \circ \tilde{\varphi}(\lambda_i)$ (for $i \in \mathbb{Z}_3$).

A triple $(\varphi_1, \varphi_2, \varphi_3) \in J^3$ of angles is called *admissible*, if there exists an admissible constellation with parameter triple $(\tilde{\varphi}^{-1}(\varphi_1), \tilde{\varphi}^{-1}(\varphi_2), \tilde{\varphi}^{-1}(\varphi_3))$.

An admissible triple is called *generic* if there is a corresponding admissible constellation such that the lines are not contained in parallel planes. The triple is called *parallel* otherwise.

Remark 5.2. Note that a general triple of three oriented lines is determined (up to the action of $SO(3)$) by the oriented distances and the angles. We have the restriction that the distance and angle match, i.e. every pair of lines can be put into a suitable helicoid.

We define $\mathcal{T}$ to be the set of interior points of the tetrahedron with vertices $(\pi, 0, 0)$, $(0, \pi, 0)$, $(0, 0, \pi)$, (π, π, π).

Sketches of admissible constellations can be found in [Bal03].

From Corollary 3.5, we have:

Lemma 5.3. *For any symmetric trinoid $M \in \mathcal{M}$ with $\tilde{\varphi}(\lambda_i) \notin \pi\mathbb{Z}$ for $i \in \{1, 2, 3\}$, the triple $\Psi(M)$ is admissible.* □

Theorem 5.4. *A triple $(\varphi_1, \varphi_2, \varphi_3) \in J^3$ is a generic admissible triple if and only if $(r(\varphi_1), r(\varphi_2), r(\varphi_3)) \in \mathcal{T}$. For every triple of that kind, there are exactly two generic admissible constellations of lines in $\mathbb{R}^3$ (modulo $SO(3)$).*

Proofs can be found in [Dan03, Prop. 9]; and [Bal03]. Essentially, the conditions on the angles correspond to the condition that the directions of the lines form a spherical triangle (after identifying the unit tangent spheres of $\mathbb{R}^3$ via parallel translation).

Proof of Theorem 1.2. The theorem follows from Lemma 5.3 and Theorem 5.4. □

Remark 5.5. One can show that an admissible triple corresponds *either* to generic *or* to parallel constellations, and that the triple of reduced angle lies in the boundary of $\mathcal{T}$ in the parallel case, see [Bal03]. Hence the name *generic* is justified.

6 Comparing to related results

In this section, we compare the conditions obtained by [Dan03] and our results with the results in in [BPS03] and [UY00].

Consider the presentation of catenoid cousins in [Bry87, Ex. 2]. Bryant parametrizes catenoid cousins with a parameter $-\frac{1}{2} < \mu_B \neq 0$.

Lemma 6.1. *The catenoid cousin given by Bryant's parameter μ_B is C_λ, where $\lambda = \tilde{\varphi}^{-1}(\pi(2\mu_B + 1))$.*

Proof. Bryant computes the total curvature of a catenoid cousin to be $-4\pi(2\mu_B + 1)$. A standard catenoid has total curvature -4π. Since a Bryant surface is locally isometric to its minimal cousin, a catenoid cousin C_λ has total curvature $-4\pi \cdot \frac{1}{\sqrt{1+4\lambda}}$ (see Example 2.3). The claim follows. $\square$

Next, we trace back the relationship between our parameters and the parameters in [BPS03].

In [BPS03, sec. 4], catenoid cousins are parametrized by a parameter $0 < \lambda_{BPS} \neq \frac{1}{2}$. Comparing the formulas for catenoid cousins given by Bryant and [BPS03], we obtain $\lambda_{BPS} = \mu_B + \frac{1}{2}$; hence, the catenoid cousin described by the parameter λ_{BPS} is C_λ, where

$$\lambda = \lambda(\lambda_{BPS}) = \tilde{\varphi}^{-1}(2\pi\lambda_{BPS}). \tag{3}$$

They consider $|\{\lambda_{BPS,i}\}|$, where $\{\cdot\}$ stands for the fractional part of a number in $[-\frac{1}{2}, \frac{1}{2})$.

The main result in [BPS03] is:

Theorem 6.2 ([BPS03, Prop. 2]). *For given parameters p_i, q_i, where $i \in \{0, 1, \infty\}$, in the generic case, it is necessary for the existence of a trinoid that the numbers $\Delta_i := |\{\lambda_{BPS,i}\}|$ satisfy the conditions*

$$\begin{aligned} \Delta_0 + \Delta_1 + \Delta_\infty &> \frac{1}{2} \\ \Delta_0 + \Delta_1 - \Delta_\infty &< \frac{1}{2} \\ \Delta_0 - \Delta_1 + \Delta_\infty &< \frac{1}{2} \\ -\Delta_0 + \Delta_1 + \Delta_\infty &< \frac{1}{2}; \end{aligned}$$

This condition is sufficient if furthermore, certain holomorphic spinors P and Q have no common zeroes. $\square$

They also show that their classification is equivalent to [UY00, Thm. 2.6].

Observe that the "generic case" in [BPS03] means that the case of half-integer $\lambda_{BPS,i}$ is excluded (cf. formula (6.3), and the remark at the bottom of page 18), so their class of generic trinoids is slightly larger than ours.

In [Dan03, Thm. 49], the trinoids from the classification of Umehara-Yamada (or equiv. Bobenko et al.) are constructed via minimal surfaces bounded by a generic constellation of three lines.

In view of (3), we find that $|\{\lambda_{BPS}\}|$ corresponds to our notion of reduced angle, i.e. we have $r(\tilde{\varphi} \circ \lambda(\lambda_{BPS})) = 2\pi|\{\lambda_{BPS}\}|$. So the necessary conditions of Theorem 6.2 are the same as those in Theorem 1.2.

Comparing our Theorem 1.2 to the main theorem of [BPS03], the condition about the common zeroes of P, Q is preventing our condition from being sufficient. In [Dan03], this additional condition is that his polynomial "φ" of degree two has no double root (for the equivalence, see [Dan03, proof of L. 16, and page 31]). This condition avoids singular points on the minimal surface and the trinoid.

Hence, Corollary 1.4 follows.

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